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Is the singularity at separation removable?

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Many numerical integrations support Goldstein's theory of the structure of the solution of a laminar boundary layer near the point of separation O when the mainstream is prescribed, and in particular confirm that the solution is singular there. The existence of the singularity, however, implies that the hypotheses of the boundary layer break down in the neighbourhood of O, and it has been suggested that the disturbance to the mainstream near O is sufficient to smooth out the singularity and enable the solution to pass over into another conventional boundary layer downstream of O containing a region of reversed flow. The aim of this paper is to explore this possibility in detail using the methods of the triple-deck, developed by the author and others, which have proved successful in somewhat related problems.

Granted the hypothesis that the interaction between the boundary layer and the mainstream is significant near separation and manifests itself through a triple deck, it is found that its streamwise extent is $O(\epsilon^2 l)$ where ϵ^{-8} is a characteristic Reynolds number, $\epsilon \ll 1$, and l a characteristic length of the problem. The upper deck is of width $O(\epsilon^2 l)$, lies entirely outside the boundary layer, and in it the flow is inviscid. The main deck is of width $O(\epsilon^4 l)$ and constitutes the majority of the boundary layer near O, and the perturbations in the velocity are largely inviscid. Finally, the lower deck is of lateral extent $O(\epsilon^{\frac{9}{2}} l)$ and is controlled by a linear equation of boundary-layer type. The whole structure is found to be consistent provided a certain integro-differential equation can be solved, which takes different forms according as the mainstream is supersonic or subsonic. When the mainstream is subsonic it is found that there is no solution to this equation that is sufficiently smooth on the downstream side of the triple deck. When the mainstream is supersonic it is found that the triple deck can at best postpone the breakdown of the assumed structure which still must occur within a distance $O(\epsilon^2 l)$ of O.

It is concluded that the singularity is not removable by the methods proposed and it is inferred that the singularity is a real phenomenon terminating the flow which, at high Reynolds number, exists upstream of O.

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1. Introduction

I have two especial favourites among the many papers that Sydney Goldstein has written. The first appeared in 1930, and in it he set out a model procedure for handling boundary-layer problems which has been followed literally by hundreds of authors right up to the present day. This is to assume that the stream function ψ can be expanded as a series of powers of x, ascending or descending, which need not be integers, whose coefficients are functions of y/x^n, where n is constant. Further the paper includes a beautiful discussion of the flow near the trailing edge of a flat plate and introduces the notion of matched asymptotic expansions, logarithms and all, to my knowledge for the first time, and certainly antedating by many years the upsurge of interest in the method which occurred during the 1950's.

The second paper, on separation, appeared in 1948 and was partly inspired by the earlier paper and partly by the work of his former student Leslie Howarth (1938). In it Goldstein showed for the first time that there is a basis for a theoretical explanation of the phenomenon of separation which had hitherto been a complete mystery to mathematicians interested in fluid mechanics. The reader is reminded that the phenomenon occurs in the high Reynolds number flow of a fluid past a bluff body and the problem is to explain why and how the mainstream breaks away from the boundary at or very near the point O where the skin friction τ vanishes. Upstream of separation the mainstream is close to the boundary, the intervening space being occupied by a thin boundary layer. Although reversed flow occurs downstream of O there is no obvious reason why it should not be accommodated in the thin boundary layer, so that there should be little change in gross flow properties on passing through O. This conclusion, of a superficial theoretical study, is contradicted by observation and also by Goldstein's deeper study. He showed that, in general, the boundary layer develops a singularity at O and that the flow immediately downstream of O cannot be joined on to the flow just upstream on the basis of boundary-layer theory alone. This is in line with the observed flow pattern, that the boundary layer ceases to exist at or near O. For convenience this paper of Goldstein will subsequently be denoted by I, the equation number following in brackets. Since it was written much further work has been done by a variety of workers, from both analytical and numerical points of view, and a review has recently been written by Brown & Stewartson (1969) to which the reader is referred for details.

The majority of the further work has been done within the frame-work of boundary-layer theory, and so far little attempt has been made to consider, on a quantitative basis, the effect of the singularity on the external flow. One of the questions raised by Goldstein in I is 'does a singularity always occur except for certain special pressure distributions near separation, and are experimental pressure distributions always of the special type?' Specifically, since the cross-velocity is infinite at separation according to boundary-layer theory, the flow induced by the boundary layer in the mainstream is not necessarily small in the neighbourhood of O even though it is negligible elsewhere. Might it not be possible for the mainstream to be so modified near O that the singularity in the boundary layer is removed and the solution enabled to pass smoothly into the reversed flow region downstream of O? Some authors have contended in fact that since the singularity has never been observed some such interaction must occur, as a result of which the mainstream is always able to adjust itself to prevent the singularity from developing. On the other hand the observations are made at finite

Reynolds number R and the theory does not predict that τ is singular at separation for finite R. Instead it predicts that

$$\lim_{R\to\infty} \tau R^{\frac{1}{2}}$$

is a function of x which becomes singular at separation—quite a different thing.

Two examples have recently been given of evolving boundary layers which pass through separation without a singularity. Catherall & Mangler (1966) studied an incompressible flow in which the pressure is linearly related to the displacement thickness near separation and found the flow to be completely regular at separation. Stewartson & Williams (1969) also found a regular solution in their study of the free-interaction zone of shock-wave boundary-layer interaction. However, the relation assumed by Catherall & Mangler is rather artificial and contrived, while the free interaction problem describes the rapid development of the boundary layer from the Blasius form and the conventional boundary-layer equations do not apply.

It is of interest of consider a classical solution of the boundary layer which is evolving towards a singularity at separation, and to examine whether the consequent sharp rise in the gradient of the displacement thickness induces changes in the pressure gradient which effectively smooth out the singularity and enable the boundary layer to develop, smoothly, into a region of reversed flow. In the immediate neighbourhood of separation (at O) we shall not therefore assume that the boundary-layer hypotheses apply, but that upstream and downstream of this neighbourhood the flow does satisfy conventional boundary-layer requirements, i.e. that streamwise variations are much smaller than the crosswise variations and viscous forces balance inertia forces. Thus the Goldstein asymptotic structure of the boundary layer is now to be regarded as an outer expansion holding when the upstream distance from O is small on the inviscid scale but large when compared with some negative power of the Reynolds number. In order to describe the flow near O we shall use an adaptation of the triple-deck structure already used successfully in a number of problems wherein the boundary layer has to react quickly to changes in boundary conditions (Stewartson 1969; Messiter 1970; Brown & Stewartson 1970; Stewartson 1970) or induces rapid changes in itself (Stewartson & Williams 1969). The form of the triple deck was in fact outlined in Goldstein's 1930 paper.

As originally conceived the decks had each a streamwise extent $O(\epsilon^3)$, where

$$\epsilon = R^{-\frac{1}{8}}, \tag{1.1}$$

and thicknesses $O(\epsilon^5)$, $O(\epsilon^4)$, $O(\epsilon^3)$, respectively in the direction normal to the boundary. In the main deck [of thickness $O(\epsilon^4)$, and therefore substantially the same as the basic boundary layer] the velocity perturbations are largely inviscid and $O(\epsilon)$ while the pressure variation is smaller being only $O(\epsilon^2)$. Such velocity variations induce a cross-velocity $O(\epsilon^2)$ at the outer edge of the boundary layer which is the source of pressure variations $O(\epsilon^2)$ in the upper deck [of thickness $O(\epsilon^3)$, lying outside the boundary layer]. In addition the velocity perturbation in the main deck leads to a velocity of slip at the inner edge near the boundary which is smoothed out in the lower deck [of thickness $O(\epsilon^5)$] by an inner boundary layer in which viscous forces are important. Further the pressure changes induced in the upper deck are transmitted largely without change through to the lower deck. The balance between the slip velocity, pressure gradient and changes in boundary conditions (if any) in the lower deck controls the whole structure of the interaction region.

One of the problems in which the triple deck plays a decisive role is that of self-induced separation in the phenomenon of shock-wave boundary-layer interaction (Stewartson & Williams 1969). Here the boundary-layer profile is originally a Blasius profile that spontaneously develops an adverse pressure gradient which induces separation, without, however, a singularity, so that it proved possible to continue the numerical integration into the reversed flow region beyond.

The present problem is rather different in that there is already an adverse pressure gradient acting in the boundary layer upstream of the separation neighbourhood, and the basic profile which is modified in the triple deck is virtually a separation profile with vanishingly small skin friction. These differences turn out to be decisive and we shall find that the application of parallel arguments to those in the self-induced separation do not enable us to remove completely the singularity at separation. Of course this does not preclude, necessarily, alternative structures but it is not obvious what they could be.

Specifically the parallel arguments lead us to the conclusion that if there is a satisfactory triple-deck structure, linking two regions where conventional boundary-layer theory may be applied, its streamwise extent is $O(\epsilon^2)$, the increase over the earlier scaling being due to the smaller skin friction in the basic boundary layer. The cross-thickness of the decks are respectively $O(\epsilon^{\frac{9}{2}})$, $O(\epsilon^4)$, $O(\epsilon^2)$ but the general principles, outlined above, governing the solution in each region remain the same. We shall find that if x measures distance downstream from the separation point, on the scale of the triple deck, computed according to conventional boundary-layer theory with a prescribed pressure gradient, and $A_1(x)$ is proportional to the skin friction at the wall, then

$$A_1^2(x) + 8\alpha_1^2 x = -\lambda \int_x^\infty \frac{A_1''(\xi)\,d\xi}{(\xi - x)^{\frac{1}{2}}}, \tag{1.2}$$

when the mainstream velocity is subsonic, while

$$A_1^2(x) + 8\alpha_1^2 x = \lambda \int_{-\infty}^x \frac{A_1''(\xi)\,d\xi}{(x - \xi)^{\frac{1}{2}}}, \tag{1.3}$$

when the mainstream velocity is supersonic. Here α_1 and λ are positive constants.

According to Goldstein's (1948) theory, $A_1(x) = 2\alpha_1(-2x)^{\frac{1}{2}}$ if $x < 0$ and there is no solution if $x > 0$, so that the right-hand sides of (1.2), (1.3) represent the effect of the interaction with the mainstream. In both cases arguments are given to show that the existence of an acceptable form for A_1 when $x \ll -1$ precludes the existence of A_1 for all x. It is inferred that the singularity is real in the following sense. Let the external pressure gradient upstream of separation be smooth and of a kind to provoke a singularity at separation (so that $\alpha_1 \neq 0$) and let the external pressure gradient downsteam of separation also be smooth. Then the feedback from the boundary layer to the mainstream in the neighbourhood of separation is not of a kind to smooth out the singularity and to enable the boundary layer to pass into an equivalent form but with reversed flow.

We conclude that, when the mainstream velocity is subsonic at separation and the boundary is thermally insulating, the flow cannot remain analytic near separation, in the boundary-layer mainstream sense, as $R \to \infty$. The same is true when the mainstream velocity is supersonic, unless the boundary layer separates spontaneously by a free interaction, at a point where just upstream the skin friction is finite. Otherwise it is most likely that the structure of the flow near separation is pathological as $R \to \infty$, which makes the task of unravelling it rather daunting.

2. Goldstein's solution near separation

Suppose that the fluid is in two-dimensional motion past a flat plate and, to begin with, let it be incompressible. Choose a system of co-ordinates Ox^*y^* with O at the point on the plate where the skin friction vanishes according to the usual boundary-layer theory. Let Ox^* be directed along the plate in the direction of the mainstream and Oy^* be directed normal to the plate and into the fluid. Further suppose that the mainstream velocity is $U^*(x^*)$, in the limit $\nu \to 0$, where ν is the kinematic viscosity and U^* is a smooth function of x^*. Specifically $\dot{U}^*$ and its first derivative exist and are continuous in the neighbourhood of $x^* = 0$ while its second derivative exists in the limit $x^* \to 0-$. Following I(13) we define a representative length l and Reynolds number R by

$$l = -\frac{U^*(0)}{U^{*\prime}(0)}, \quad R = \frac{U^*(0)\,l}{\nu}, \tag{2.1}$$

primes denoting differentiation with respect to x^*, and $U^{*\prime}(0) < 0$, since we need an adverse pressure gradient to provoke separation in the first place. Then, if (u^*, v^*) are the velocity components in the (x^*, y^*) directions, the boundary-layer equations, which need to be solved to obtain Goldstein's solution near separation, are

$$\left.\begin{aligned} u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} &= U^*\frac{dU^*}{dx^*} + \nu\frac{\partial^2 u^*}{\partial y^{*2}}, \\ \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} &= 0, \end{aligned}\right\} \tag{2.2}$$

with boundary conditions

$$u^* = v^* = 0 \quad \text{at} \quad y^* = 0, \quad u^* \to U^* \quad \text{as} \quad y^* \to \infty, \tag{2.3}$$

together with some appropriate initial condition at a negative value of x^*, for example at the leading edge of the plate.

Goldstein showed that as $x^* \to 0-$, i.e. just upstream of separation, a solution of these equations could be found in the following double structural form. First, there is an inner region where

$$y^* = O[lR^{-\frac{1}{2}}(-x^*/l)^{\frac{1}{4}}], \tag{2.4}$$

in which we define new variables

$$\xi = (-x^*/l)^{\frac{1}{4}}, \quad \eta = R^{\frac{1}{2}} y^*/2^{\frac{1}{2}} l \xi \tag{2.5}$$

and write
$$u^* = 2U^*(0)\,\xi^2 [f_0'(\eta) + \xi f_1'(\eta) + \xi^2 f_2'(\eta) + \ldots], \tag{2.6}$$

with a corresponding form for v^*. Here the f_n are functions of η only, primes denote differentiation with respect to η and the series contains integer powers of ξ and (eventually) $\log \xi$ (Stewartson 1958). The leading functions f_n are

$$f_0 = \tfrac{1}{6}\eta^3, \quad f_1 = \alpha_1 \eta^2, \quad f_2 = \frac{2^{\frac{1}{4}}\pi^{\frac{3}{2}}\alpha_1^2}{5(\frac{1}{4}!)^3}\eta^2 - \frac{\alpha_1^2}{15}\eta^5, \tag{2.7}$$

α_1 being an arbitrary constant, presumably dependent in some way on the initial conditions, on u^* and on the mainstream velocity when $x^* < 0$. The further f_n depend on the solution of linear differential equations with increasingly complicated forcing terms; some of them have been computed by Jones (1948) and Terrill (1960).

Secondly, there is an outer region where $y^* > 0$ in which we define

$$y = R^{\frac{1}{2}} y^*/l \tag{2.8}$$

and write
$$u^* = U^*(0)\left[U_0(y) + 2^{\frac{3}{2}}\alpha_1 \xi^2 U_0'(y) + \frac{2^{\frac{7}{4}}\pi^{\frac{3}{2}}}{5(\frac{1}{4}!)^3}\alpha_1^2 \xi^3 U_0'(y) - \frac{2^{\frac{3}{2}}\pi}{(\frac{1}{4}\,!)^2}\alpha_1^3 \xi^4 \log \xi\, U_0'(y) + \xi^4 U_4(y) + \ldots\right]. \tag{2.9}$$

Here $U_0(y)$ is a function of y and U_4 may be expressed in terms of U_0, I(130). The function $U_0(y)$ is actually the separation profile and is largely dependent on the boundary conditions imposed upstream of O (2.3). However, it does have an expansion about $y = 0$ in integer powers of y and $\log y$, of which any non-zero power of $\log y$ also contains a power of y equal to at least *eight* (Brown & Stewartson 1969). It begins

$$U_0(y) = \tfrac{1}{2}y^2 - \tfrac{1}{6}\alpha_1^2 y^4 + \tfrac{3}{2}\sqrt{2}\,A_{30} y^5 + \tfrac{7}{4}A_{40} y^6 + \ldots, \tag{2.10}$$

where A_{30}, A_{40} are defined in I(111) and I(113) respectively,† and $U_0(y) \to 1$ as $y \to \infty$.

It is noted that, since f_0, f_1, f_2 are polynomials, a single unified expansion can be written down which is valid when both (2.6) and (2.9) hold provided terms in u^* of order ξ^4 are neglected. On retaining higher powers of ξ, however, the properties of f_3, etc., preclude such a simple unification and there seems no real advantage to be derived.

Of particular importance is the behaviour of v^* in the outer region where $y > 0$, i.e.

$$v^* = \frac{R^{-\frac{1}{2}}U^*(0)}{2\xi^2}\left[\alpha_1 2^{\frac{3}{2}} U_0(y) + \frac{3\,.\,2^{\frac{3}{4}}\pi^{\frac{3}{2}}}{5(\frac{1}{4}!)^3}\alpha_1^2 \xi U_0(y) + O(\xi^2 \log \xi)\right], \tag{2.11}$$

revealing that, as $y \to \infty$,

$$v^* \to \frac{R^{-\frac{1}{2}}\,U^*(0)\,\alpha_1 (2l)^{\frac{1}{2}}}{(-x^*)^{\frac{1}{2}}}\left\{1 + O\left[\left(-\frac{x^*}{l}\right)^{\frac{1}{4}}\right]\right\} \tag{2.12}$$

and has a clear singularity as $x^* \to 0-$.

† The constants A_{30}, A_{40} are called A_3, A_4 in I, but we shall need A_3, A_4 for functions of x in the triple deck.

Now the cross-velocity at the outer edge of a boundary layer induces a secondary flow in the mainstream outside, but since it is usually $O(R^{-\frac{1}{2}})$ it does not significantly alter the character of the mainstream flow. However, near separation, (2.12) shows that this cross-velocity is large and tends to infinity as $x^* \to 0-$ so that its effect on the mainstream can no longer be neglected. Hence an interaction must occur between the mainstream and the boundary layer, and hopefully the net result is that the cross-velocity, while large in comparison with $R^{-\frac{1}{2}}$, remains finite at separation and the singularity disappears. It is not immediately clear from (2.12) what is the streamwise scale, over which this interaction could take place, and it is actually found in the end from the condition that *in principle* a consistent expansion can be set up. In order to save space we shall *assume* that the appropriate streamwise scale is ϵ^2 and demonstrate *a posteriori* that it is consistent in principle. The scaling turns out to be unique if the interaction is through a triple deck and we shall then show that the resulting equations do not lead to an acceptable solution. The inference is drawn that no scaling whatever can remove the singularity in (2.12).

Let us write

$$\left.\begin{aligned} &x^* = l\epsilon^2 x, \quad u^* = U^*(0)\, u(x,y,\epsilon), \quad v^* = U^*(0)\, v(x,y,\epsilon), \\ &p^* = p_0 + \rho U^{*2}(0)\, p(x,y,\epsilon), \quad \epsilon^8 = R^{-1}, \end{aligned}\right\} \tag{2.13}$$

where y is defined in (2.8), p^* is the pressure, p_0 its value at O and ρ is the density of the fluid. Then Goldstein's expansion (2.9) tells us that, when x is large and negative and

$$y \gg \epsilon^{\frac{1}{2}}|-x|^{\frac{1}{4}}, \quad \text{i.e.} \quad \eta \gg 1, \tag{2.14}$$

$$u \sim U_0(y) + 2^{\frac{3}{2}}\alpha_1(-x)^{\frac{1}{2}}\epsilon U_0'(y) + \frac{2^{\frac{7}{4}}\pi^{\frac{3}{2}}}{5(\frac{1}{4}!)^3}\alpha_1^2\epsilon^{\frac{3}{2}}(-x)^{\frac{3}{4}}\, U_0'(y) + \ldots \tag{2.15}$$

with an equivalent form for v. It is clear from (2.14), (2.15) that the limit $x \to -\infty$ is non-uniform. What we have in mind is that (2.15) holds when $-1 \gg x \gg -\epsilon^{-2}$; (2.14) then implies that $y > 0$.

Suppose now that x is large and negative but η is finite. Then from (2.6) we see that

$$u \sim 2(-x)^{\frac{1}{2}}\epsilon\{f_0'(\eta) + \epsilon^{\frac{1}{2}}(-x)^{\frac{1}{4}}f_1'(\eta) + \epsilon(-x)^{\frac{1}{2}}f_2'(\eta) + \ldots\}, \tag{2.16}$$

where we now set

$$y = \epsilon^{\frac{1}{2}}z, \quad \eta = z/(-4x)^{\frac{1}{4}}. \tag{2.17}$$

Again the limit $x \to -\infty$ is non-uniform and it should be understood that $-1 \gg x \gg -\epsilon^{-2}$ and η is finite.

The scaling of the lower deck implied by (2.17) is different from that in previous studies but is natural once we have agreed that the streamwise extent of the triple deck is $O(\epsilon^2)$.

3. The triple deck

The point of view we explore now is that the effect of the formal singularity at $x^* = 0$ in Goldstein's solution is actually confined to the immediate neighbourhood of $x^* = 0$, the boundary-layer hypotheses with a smooth mainstream holding elsewhere, and that, when the interaction with the mainstream is taken into account, the flow remains analytic near $x^* = 0$. Further, we suppose that the interaction takes the form of a triple deck of streamwise extent $O(\epsilon^2)$. Quantifying this line of argument we now assume that when $x = O(1)$, $y = O(1)$, which constitutes the major part of the boundary-layer region and which we call the main deck, the solution of the Navier–Stokes equations may be expressed in the form

$$u = U_0(y) + \epsilon u_1(x, y) + \epsilon^{\frac{3}{2}} u_2(x, y) + \epsilon^2 \log \epsilon \, u_3(x, y) + \epsilon^2 u_4(x, y) + \ldots, \quad (3.1a)$$

$$v = \epsilon^3 v_1(x, y) + \epsilon^{\frac{7}{2}} v_2(x, y) + \epsilon^4 \log \epsilon \, v_3(x, y) + \epsilon^4 v_4(x, y) + \ldots, \quad (3.1b)$$

$$p = \epsilon^2 p_2(x, y) + \epsilon^3 p_3(x, y) + \ldots. \quad (3.1c)$$

The structure (3.1) is partly suggested by the behaviour of the boundary layer as $x \to -\infty$, which was found in I, and partly by the known properties of the inviscid flow field outside the triple deck. The boundary conditions on u_n as $x \to -\infty$ follow from (2.15) and are

$$(-x)^{-\frac{1}{2}} u_1(x, y) \to 2^{\frac{3}{2}} \alpha_1 U_0'(y), \quad (-x)^{-\frac{3}{4}} u_2 \to \frac{2^{\frac{7}{4}} \pi^{\frac{3}{2}}}{5(\frac{1}{4}!)^3} \alpha_1^2 U_0'(y),$$
$$(-x)^{-1} u_3(x, y) \to -\frac{2^{\frac{1}{2}} \pi}{(\frac{1}{4}!)^2} \alpha_1^3 U_0'(y), \text{ etc.}, \quad (3.2)$$

with corresponding results for v_n. In addition

$$(-x)^{-1} p_2 \to -1, \quad (-x)^{-\frac{3}{2}} p_3 \to 0 \quad (3.3)$$

as $x \to -\infty$ because of the known properties of the pressure upstream of $x^* = 0$. It is noted that the expansion for the pressure begins with a higher-order term than that for u and this is a characteristic feature of the main deck. It means that the primary variations in velocity are formally independent of pressure and are controlled only by what happens below the main deck.

The expansion (3.1) is now substituted into the full Navier–Stokes equations and sets of equations for the various unknown functions are obtained by comparing coefficients of successive powers of ϵ in ascending order. The leading equations are

$$\left.\begin{aligned} U_0(y) \frac{\partial u_n}{\partial x} + v_n \frac{dU_0}{dy} = 0, \quad \frac{\partial u_n}{\partial x} + \frac{\partial v_n}{\partial y} = 0 \quad (n = 1, 2, 3) \\ \text{and} \qquad \partial p_n / \partial y = 0 \quad (n = 2, 3), \end{aligned}\right\} \quad (3.4)$$

confirming that the principal variations in the main deck are inviscid. The viscous terms in the Navier–Stokes equations only enter the equations for u_4, v_4 and subsequently; these terms also reflect the fact that, even outside the neighbourhood of the singularity, $v^* \neq 0$ and u^* varies with x^*. The equations (3.4) are the same as those considered by Goldstein and we have

$$u_n = A_n(x) \, U_0'(y), \quad v_n = -A_n'(x) \, U_0(y) \quad (n = 1, 2, 3), \quad (3.5)$$

where the A_n are functions of x whose only properties, so far known, are that they conform with (3.2) as $x \to -\infty$. The equations for u_4, v_4 were also considered in I(130) and again the solutions are the same, except that the explicit functions of

x given by Goldstein should be replaced by general functions of x but which are the same as his in the limit $x \to -\infty$. Thus we find that

$$U_0^2(y)\frac{\partial}{\partial y}\left(\frac{v_4}{U_0(y)}\right) = \frac{dp_2}{dx} - U_0''(y) + A_1(x)\,A_1'(x)\,\{U_0'^2 - U_0\,U_0''\}, \tag{3.6}$$

the term $U_0''(y)$ signifying the first appearance of the viscous terms in the expansion. Hence

$$v_4(x,y) = -U_0(y)\,A_4(x) - A_1(x)\,A_1'(x)U_0'(y) + U_0(y)\int_y^\infty \frac{U_0''(y_1)}{U_0^2(y_1)}\,dy_1 + U_0(y)\frac{dp_2}{dx}\left\{y - \int_y^\infty\left(\frac{1}{U_0^2(y_1)} - 1\right)dy_1\right\}, \tag{3.7}$$

where A_4 is a function of x to be found, and which tends to the constant limit given in I(131) as $x \to -\infty$. The equation of continuity may now be used to deduce u_4. As $y \to \infty$

$$v_4 - y\,dp_2/dx \to -A_4(x) \tag{3.8}$$

and when y is small

$$v_4 = \frac{2}{3y}\left[1 - \frac{dp_2}{dx}\right] + \text{higher powers of } y. \tag{3.9}$$

We shall see below in (3.21), that $dp_2/dx \equiv 1$, and, assuming it,

$$v_4 = -(4\alpha_1^2 + A_1A_1')\,y - 60A_{30}\,2^{\frac{1}{2}}y^2\log y + O(y^2) \tag{3.10}$$

when y is small.

Further terms may be worked out if desired without any formal difficulty, but these are sufficient for our purposes. We now consider the structure of the main deck as $y \to \infty$. Since $U_0 \to 1$ as $y \to \infty$, we obtain immediately that

$$v - y\epsilon^4\,dp_2/dx \to -\epsilon^3A_1'(x) - \epsilon^{\frac{7}{2}}A_2'(x) - \epsilon^4\log\epsilon A_3'(x) - \epsilon^4A_4'(x) + o(\epsilon^4) \tag{3.11}$$

while

$$u \to -\epsilon^2p_2(x) + o(\epsilon^2). \tag{3.12}$$

The normal velocity at the outer edge of the main deck induces an inviscid perturbation in the mainstream. To find its properties we set up an upper deck in which the length scales are the same in the x^* and y^* directions. Hence we introduce a new co-ordinate Y satisfying

$$\epsilon^2 y = Y \tag{3.13}$$

and, further, write

$$\left.\begin{aligned} u &= 1 + \epsilon^2\hat{u}_2(x,Y) + \epsilon^3\hat{u}_3(x,Y) + \ldots, \\ v &= \epsilon^2\hat{v}_2(x,Y) + \epsilon^3\hat{v}_3(x,Y) + \ldots, \\ p &= \epsilon^2\hat{p}_2(x,Y) + \epsilon^3\hat{p}_3(x,Y) + \ldots. \end{aligned}\right\} \tag{3.14}$$

Then, on substituting into the Navier–Stokes equations and equating powers of ϵ we find that, as expected, the equations governing $\hat{u}_2$, $\hat{v}_2$, $\hat{p}_2$ and $\hat{u}_3$, $\hat{v}_3$, $\hat{p}_3$ are inviscid and linear, being

$$\frac{\partial\hat{u}_n}{\partial x} = -\frac{\partial\hat{p}_n}{\partial x}, \quad \frac{\partial\hat{v}_n}{\partial x} = -\frac{\partial\hat{p}_n}{\partial Y}, \quad \frac{\partial\hat{u}_n}{\partial x} + \frac{\partial\hat{v}_n}{\partial Y} = 0 \quad (n = 2,3). \tag{3.15}$$

In addition, from (3.11),

$$\hat{p}_2 = p_2(x), \quad \hat{p}_3 = p_3(x), \quad \hat{v}_2 = 0, \quad \hat{v}_3 = -A_1'(x) \quad \text{at} \quad Y = 0 \tag{3.16}$$

and $$\hat{p}_2 \to x, \quad (-x)^{-\frac{3}{2}}\hat{p}_3 \to 0 \quad \text{as} \quad x \to -\infty. \tag{3.17}$$

The conditions to be imposed on $\hat{p}_n$ as $x \to \infty$, i.e. as we leave the triple deck on the downstream side, are not of course known except that the boundary layer is then assumed not to be pathological. At worst therefore we could have $\hat{p}_3 \sim x^{\frac{3}{2}}$. If $\hat{p}_3 \to 0$ when $|x| \to \infty$ then it follows from (3.15), (3.16) that

$$\hat{p}_3(x, Y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{A_1'(x_1)(x-x_1)}{(x_1-x)^2+Y^2}\,dx_1, \tag{3.18}$$

so that $$p_3(x) = p_3(x,y) = \hat{p}_3(x,0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{A_1'(x_1)\,dx_1}{x-x_1}. \tag{3.19}$$

This formula can also be used to cover the possibility that $p_3 \sim |x|^{\frac{3}{2}}$ as $|x| \to \infty$ by using it in the twice-differentiated form

$$p_3''(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{A_1'''(x_1)\,dx_1}{x-x_1}. \tag{3.20}$$

The restrictions on $\hat{p}_2$ imposed by (3.15), (3.16), (3.17) however are sufficient to define it to be uniquely

$$\hat{p}_2(x, Y) \equiv x \quad \text{so that} \quad p_2(x) = x, \tag{3.21}$$

a result anticipated earlier in (3.10).

We now turn to the behaviour of the main deck as $y \to 0$. From (2.10), (3.5), (3.10) we have, when y is small,

$$u = [\tfrac{1}{2}y^2 - \tfrac{1}{6}\alpha_1^2 y^4 + O(y^5)] + \epsilon[y - \tfrac{2}{3}\alpha_1^2 y^3 + \ldots]A_1(x) + \ldots + \epsilon^{\frac{3}{2}}[y - \ldots]A_2(x)$$
$$+ \epsilon^2 \log\epsilon[y - \ldots]A_3(x) + [\tfrac{1}{2}A_1^2(x) + 4\alpha_1^2 x + \ldots]\epsilon^2, \tag{3.22}$$
$$v = \epsilon^3[-\tfrac{1}{2}y^2 - \ldots]A_1'(x) + \ldots, \tag{3.23}$$
$$p = \epsilon^2 x + \epsilon^3 p_3(x) + \ldots. \tag{3.24}$$

At the plate $y = 0$ we must apply the no-slip condition which, however, is not satisfied by (3.22) in the term $O(\epsilon^2)$. Hence we introduce an inner boundary layer, or lower deck, to adjust the value of u to zero, the driving mechanism being the pressure gradient. It is found by trial that the appropriate scaling is given by (2.17), namely

$$y = \epsilon^{\frac{1}{2}}z. \tag{3.25}$$

Then the boundary condition which u must satisfy in the lower deck as $z \to \infty$ is that

$$u \approx \epsilon[\tfrac{1}{2}z^2] + \epsilon^{\frac{3}{2}}[zA_1(x)] + \epsilon^2[zA_2(x) - \tfrac{1}{6}\alpha_1^2 z^4 + \tfrac{1}{2}A_1^2(x) + 4\alpha_1^2 x] + \ldots \tag{3.26}$$

from (3.22), with a corresponding form for v. This suggests that we write, in the lower deck,

$$\left.\begin{aligned} u &= \tfrac{1}{2}\epsilon z^2 + z\epsilon^{\frac{3}{2}}A_1(x) + \epsilon^2\tilde{u}_3(x,z) + \ldots,\\ v &= -\tfrac{1}{2}z^2\epsilon^4 A_1'(x) + \epsilon^{\frac{9}{2}}\tilde{v}_3(x,z) + \ldots,\\ p &= \epsilon^2 x + 0\,.\,\epsilon^{\frac{5}{2}} + \epsilon^3\tilde{p}_3(x,z) + \ldots. \end{aligned}\right\} \tag{3.27}$$

In this expansion we have anticipated the simplicity of the two leading terms in the pressure and in u. More general forms could be assumed but the only formal possibility seems to be given by (3.27). Again on substituting into the full Navier–Stokes equations and using the relations between x, z and the physical co-ordinates we find that, formally, the expansion (3.27) is justified. Further it may now be verified that the scaling $x^* = \epsilon^2 x$ used throughout the triple deck is in fact the only possibility. Moreover, we find that $\tilde{p}_3$ is independent of z so that

$$\tilde{p}_3(x,z) = p_3(x) \tag{3.28}$$

and is defined in terms of $A_1(x)$ by (3.19) or (3.20). Finally, provided we neglect terms in u and v which are $o(\epsilon^2)$ and $o(\epsilon^{\frac{7}{2}})$, respectively, the equations satisfied by u, v are just the boundary-layer equations in terms of x, z.

On substitution it then follows that

$$\left.\begin{aligned} \tfrac{1}{2}z^2\frac{\partial \tilde{u}_3}{\partial x} + z\tilde{v}_3 + \tfrac{1}{2}z^2 A_1 A_1' &= -\frac{dp_3}{dx} + \frac{\partial^2 \tilde{u}_3}{\partial z^2}, \\ \frac{\partial \tilde{u}_3}{\partial x} + \frac{\partial \tilde{v}_3}{\partial z} &= 0. \end{aligned}\right\} \tag{3.29}$$

The boundary conditions associated with (3.29) are

$$\tilde{u}_3 = \tilde{v}_3 = 0 \quad \text{at} \quad z = 0 \tag{3.30}$$

to satisfy the no-slip condition at the plate;

$$\tilde{u}_3 - zA_2(x) + \tfrac{1}{6}\alpha_1^2 z^4 \to \tfrac{1}{2}A_1^2(x) + 4\alpha_1^2 x, \tag{3.31a}$$

$$\tilde{v}_3 + \tfrac{1}{2}z^2 A_2'(x) + A_1 A_1' z + 4\alpha_1^2 z \to 0 \quad \text{as} \quad z \to \infty. \tag{3.31b}$$

Equation (3.31) also holds in the limit $x \to -\infty$ except that A_1, A_2 take on the forms required to match with the Goldstein expansion (I, §3). This means, for example, that

$$\tfrac{1}{2}A_1^2(x) + 4\alpha_1^2 x \to 0 \quad \text{as} \quad x \to -\infty. \tag{3.32}$$

It is of interest to note that the right-hand side of (3.31*b*) is determinate. Usually in boundary-layer calculations one would expect it to be a function of x which could not be specified in advance. Here, however, if it were not zero, (3.29) would imply that $\partial\tilde{u}_3/\partial x$ contained a term proportional to z^{-1} and hence that $\tilde{v}_3$ a term proportional to $\log z$ which contradicts (3.31*b*).

In order to solve (3.29) subject to these conditions we write

$$\left.\begin{aligned} \tilde{u}_3(x,z) &= zA_2(x) - \tfrac{1}{6}\alpha_1^2 z^4 + \tfrac{1}{2}A_1^2(x) + 4\alpha_1^2 x + \tilde{\tilde{u}}_3(x,z), \\ \tilde{v}_3(x,z) &= -\tfrac{1}{2}z^2 A_2'(x) - (A_1 A_1' + 4\alpha_1^2)z + \tilde{\tilde{v}}_3(x,z), \end{aligned}\right\} \tag{3.33}$$

when $\tilde{\tilde{u}}_3$, $\tilde{\tilde{v}}_3$ satisfy

$$\tfrac{1}{2}z^2\frac{\partial \tilde{\tilde{u}}_3}{\partial x} + z\tilde{\tilde{v}}_3 = -\frac{dp_3}{dx} + \frac{\partial^2 \tilde{\tilde{u}}_3}{\partial z^2}, \quad \frac{\partial \tilde{\tilde{u}}_3}{\partial x} + \frac{\partial \tilde{\tilde{v}}_3}{\partial z} = 0, \tag{3.34}$$

with boundary conditions

$$\tilde{\tilde{v}}_3(x,0) = 0, \quad \tilde{\tilde{u}}_3(x,0) = -\tfrac{1}{2}A_1^2(x) - 4\alpha_1^2 x, \quad \tilde{\tilde{u}}_3(-\infty,z) = \tilde{\tilde{u}}_3(x,\infty) = 0. \tag{3.35}$$

As written, (3.34) may be solved by classical methods provided $p_3(x)$ and $A_1(x)$ satisfy

$$A_1^2(x) + 8\alpha_1^2 x = -\frac{(-\frac{1}{4})!}{(\frac{1}{4}!)\,2^{\frac{1}{2}}}\int_{-\infty}^{x}\frac{p_3'(x_1)\,dx_1}{(x-x_1)^{\frac{1}{4}}}. \tag{3.36}$$

Details of the solution are given in the appendix. Thus we have obtained a second relation connecting p_3 and A_1, in addition to (3.19), and are now in a position to answer the problem posed in the title of this paper, on the basis of the notion of a triple deck. We have to consider whether it is possible for an $A_1(x)$ to exist for all x, to satisfy (3.36) and (3.19), and to be a smooth function of x when $x \gg 1$, which could then be used to extend the solution into a region where the conventional equations of laminar boundary-layer theory might be expected to apply and there is a reversed flow near the plate. Before attempting to carry out this programme, however, it is convenient to consider the compressible analogue of these results.

4. Compressible boundary layers

Strictly speaking, in order to set up a rational expansion procedure for the structure of the flow near separation in a compressible fluid we should start with the full Navier–Stokes equations for such a fluid and set up a triple deck on the lines of the study made for self-induced separation (Stewartson & Williams 1969). In this paper, however, we are largely interested in the leading terms of such an expansion—indeed our aim is to show that the leading characteristic function, which is effectively $A_1(x)$, satisfies an integro-differential equation which cannot have an acceptable solution for all x. Now, provided we restrict ourselves to the leading term, it is clear, from our study above for the incompressible fluid, that the pressure gradient may be taken as independent of y, or z, throughout the main and lower decks and that there the boundary-layer equations are sufficient to describe the solution. The same remains true when the fluid is compressible.

Let us suppose that the following properties hold. First, the Prandtl number σ of the fluid is unity. Secondly, the fluid obeys a slightly modified version of the Chapman viscosity law, so that

$$\mu/\mu_0 = CT/T_0 \quad \text{with} \quad C = \mu_w T_0/\mu_0 T_w, \tag{4.1}$$

where the suffix 0 refers to conditions in the mainstream at separation, w to conditions at the wall, μ is the viscosity and T is the absolute temperature. Third, the plate is thermally insulated. Then the boundary-layer equations for a compressible fluid may be transformed into those for an incompressible fluid, it being only necessary to apply simple transformations to the x^*, y^* co-ordinates while leaving the stream function unaltered. Provided we take, as our reference state, conditions in the mainstream at separation instead of the more usual conditions at infinity, in the present instance the net effect of taking compressibility into account in the main and lower decks is to redefine ϵ as

$$\epsilon = \left\{\frac{lU^*(0)}{C\nu_0[1+\frac{1}{2}(\gamma-1)M_0^2]}\right\}^{-\frac{1}{8}}, \quad l = -\frac{U^*(0)}{U^{*\prime}(0)}, \tag{4.2}$$

leaving (3.36), (3.16) unaltered. Here M_0 is the Mach number of the mainstream at separation.

The significant change occurs in the upper deck where the small perturbations in the inviscid flow outside the boundary layer now satisfy the Prandtl–Glauert equation instead of Laplace's equation. This means that (3.19) is changed to

$$p_3(x) = \frac{-1}{\pi[1-M_0^2]^{\frac{1}{2}}}\int_{-\infty}^{\infty}\frac{A_1'(x_1)\,dx_1}{x_1-x} \quad \text{if} \quad M_0 < 1 \tag{4.3}$$

and to $$p_3(x) = -\frac{1}{[M_0^2-1]^{\frac{1}{2}}} A_1'(x) \quad \text{if} \quad M_0 > 1. \tag{4.4}$$

If the integral in (4.3) does not converge because A_1' does not vanish at $x = \infty$, then, as in (3.20), we consider the second derivative of (4.3) with respect to x.

There is no formal difficulty about relaxing the conditions imposed above on the fluid and the results obtained are not thereby significantly altered. The relaxation of the condition that there is no heat transfer across the plate at O leads analytically to the conclusion that the singularity found by Goldstein then disappears (Stewartson 1962). On the other hand careful numerical solutions by Merkin (1969) and P. G. Williams (private communication) show clear evidence that the singularity is still there. The explanation for the discrepancy between the two approaches is not yet known and so it is felt to be premature to consider the effect of heat transfer in the present context.†

5. The critical equation

In this section we consider the integro-differential equation satisfied by $A_1(x)$ and demonstrate that it cannot have a smooth solution for all x. First we consider the equation when the mainstream is subsonic so that (4.3) and (3.36) are appropriate. On eliminating $p_3(x)$ by means of (3.19), we have

$$\begin{aligned} A_1^2(x) + 8\alpha_1^2 x &= \frac{\lambda}{\pi}\int_{-\infty}^{x} \frac{dx_1}{(x-x_1)^{\frac{1}{2}}} \frac{d}{dx_1}\int_{-\infty}^{\infty} \frac{A_1'(x_2)\,dx_2}{x_2-x_1} \\ &= \frac{\lambda}{\pi}\int_{-\infty}^{x} \frac{dx_1}{(x-x_1)^{\frac{1}{2}}}\int_{-\infty}^{\infty} \frac{A_1''(x_2)\,dx_2}{x_2-x_1} \\ &= \frac{\lambda}{\pi}\int_{-\infty}^{\infty} A_1''(x_2)\,dx_2 \int_{-\infty}^{x} \frac{dx_1}{(x_2-x_1)(x-x_1)^{\frac{1}{2}}} = \lambda\int_{x}^{\infty} \frac{A_1''(x_2)\,dx_2}{(x_2-x)^{\frac{1}{2}}}, \end{aligned} \tag{5.1}$$

where $$\lambda = \frac{(-\frac{1}{4})!}{(\frac{1}{4}!)\, 2^{\frac{1}{2}}\, |1-M_0^2|^{\frac{1}{2}}} \tag{5.2}$$

and we have supposed that $A_1' \to 0$ smoothly as $x \to \infty$. This equation is critical for the success of the procedure, for the whole structure of the triple deck has been shown to be consistent and, further, may be extended to as many terms as we like provided only that this equation, in which ϵ does not appear, can be solved for $A_1(x)$.

In order to match with Goldstein's solution, when x is large and negative $A_1(x) \approx \alpha_1(-8x)^{\frac{1}{2}}$. When x is large and positive the use of (3.19) implies that $A_1' \to 0$ as $x \to \infty$. However, if that is the case (5.1) leads to a contradiction when $x \gg 1$. For, in order that the solution, as $x \to \infty$, can be joined to a conventional boundary layer we must have p_3 a smooth function of x when x is large. Thus if $p_3 \to 0$, which is a consequence of $A_1' \to 0$, then also $p_3' \to 0$, i.e. $A_1'' \to 0$ as $x \to \infty$. But if both A_1' and A_1'' vanish at infinity the right-hand side of (5.1) also vanishes at infinity contradicting the left-hand side which $\to \infty$. As mentioned earlier it is not essential for $p_3 \to 0$ in order to effect a smooth junction with a conventional

† *Note added in proof.* A formal expansion allowing for a singularity has recently been obtained by Dr J. Buckmaster provided that the plate is cooled.

boundary layer but the worst allowable behaviour is that $p_3 \sim x^{\frac{3}{2}}$ as $x \to \infty$ which implies that the pressure gradient across the two conventional boundary layers, for $x^* < 0$ and for $x^* > 0$, has an infinite second derivative at $x^* = 0+$. If that were the case, then $A_1' \sim x^{\frac{3}{2}}$ as $x \to \infty$ and the form to use, equivalent to (3.19), is (3.20), or in other words the appropriate integral equation is the derivative of (5.1) with respect to x, i.e.

$$2A_1 A_1' + 8\alpha_1^2 = \lambda \int_x^\infty \frac{A_1'''(x_2)\,dx_2}{(x_2 - x)^{\frac{1}{2}}}. \tag{5.3}$$

However, if $A_1 \sim x^{\frac{5}{2}}$ and, with its derivatives up to the third, is smooth, then the right-hand side of (5.3) must tend to a finite limit as $x \to \infty$ while the left-hand side tends to ∞, and again we have a contradiction.

It is concluded that when the mainstream is subsonic no acceptably smooth solution of (5.1) or (5.3) exists and hence that the assumption of a triple deck, joining two conventional boundary layers and smoothing out the singularity implied by the upstream one at separation, is false. Thus this relatively simple picture is inadequate, but, from a theoretical standpoint, the modification needed to describe the flow adequately near separation is not obvious. In view of the experimental evidence the most likely error in the theoretical picture lies in the assumption that there is a conventional boundary layer downstream of separation; instead the singularity is a real limit situation as $\epsilon \to 0$ and terminates the boundary layer. Downstream the flow is of quite a different character and may well be pathological. Of course no one has observed the singularity since the experiments are perforce carried out at finite Reynolds number R. The manifestation of the theoretical result is the observed dramatic break-away of the mainstream at or near separation.

Now let us consider the equation satisfied by $A_1(x)$ when $M_0 > 1$, i.e. the mainstream is supersonic at separation. Here (3.36) and (4.4) are appropriate, and on eliminating p_3 we obtain

$$A_1^2(x) + 8\alpha_1^2 x = \lambda \int_{-\infty}^x \frac{A_1''(x_1)\,dx_1}{(x - x_1)^{\frac{1}{2}}}. \tag{5.4}$$

The different forms (5.1) and (5.4) of the critical equation when the mainstream is subsonic and when the mainstream is supersonic reflect the elliptic and hyperbolic character of the governing equations in the two cases. Physically they reflect the dependence of p_3 on the overall properties of A_1 when $M_0 < 1$ and on the upstream properties only when $M_0 > 1$.

Again, when x is large and negative $A_1 \approx \alpha_1 (-8x)^{\frac{1}{2}}$ to agree with Goldstein's expansion and the right-hand side is then vanishingly small. Thus our assumption about the triple deck as an intermediate region near separation is consistent and the only question left is whether it can be extended to large positive values of x to join up with a conventional boundary layer in x^*. The numerical integration of (5.4) looks, at first sight, to be possible but we have not carried it out, partly because the computation turned out to be difficult and partly because of a surprising non-uniqueness in the solution. To see this, write

$$A_1(x) = \alpha_1(-8x)^{\frac{1}{2}} + f(x) \tag{5.5}$$

and take x to be sufficiently large and negative that squares of f may be neglected. Then

$$-f(x)+\frac{\lambda}{4\alpha_1 2^{\frac{1}{2}}(-x)^{\frac{1}{2}}}\int_{-\infty}^{x}\frac{f''(x_1)\,dx_1}{(x-x_1)^{\frac{1}{2}}}=\frac{\lambda}{4(-x)^{\frac{3}{2}}}. \tag{5.6}$$

In addition to the particular integral which can, formally, be expressed as an asymptotic series in descending powers of x with leading term $-\lambda/4(-x)^{\frac{3}{2}}$, the integro-differential equation (5.6) also possesses a homogeneous solution

$$f\sim(-x)^{-\frac{1}{12}}\sum_{n=0}^{\infty}\frac{B_n}{(-x)^{\frac{4}{3}n}}\exp[-\theta(-x)^{\frac{4}{3}}],\quad \theta=\frac{3\alpha_1^{\frac{2}{3}}}{\pi^{\frac{1}{3}}\lambda^{\frac{2}{3}}2^{\frac{1}{3}}}, \tag{5.7}$$

where B_0 is arbitrary and the remaining constants B_n are uniquely determinate in terms of B_0, α_1, λ. Thus the equation (5.4) does not have a unique solution when x is large and negative: the physical significance of this measure of arbitrariness is not known.

Although it appears that we can obtain an infinity of solutions of (5.4) for large negative x it is not possible to continue any of them to include all values of x. Suppose it can be done. Then on inverting (5.4) we have

$$A_1'(x)=\frac{1}{\lambda\pi}\int_{-\infty}^{x}\frac{A_1^2(x_1)+8\alpha_1^2x_1}{(x-x_1)^{\frac{1}{2}}}\,dx_1. \tag{5.8}$$

From (5.5), (5.6) the integrand of (5.8) is proportional to $(-x_1)^{-1}(x-x_1)^{-\frac{1}{2}}$ when x_1 is large and negative. Hence when x is large and positive the contribution to $A_1'(x)$, from that part of the range of integration in which $x_1 < 0$, is of order $x^{-\frac{1}{2}}$. On the other hand, the contribution from $x_1 > 0$ is certainly positive and greater than $(32\alpha_1^2/3\lambda\pi)\,x^{\frac{3}{2}}$. It follows that, for sufficiently large x, $A_1'(x) > 0$ and that $A_1 \to \infty$ with x. Consequently a constant $c_1(>0)$ can be chosen so that

$$A_1'(x) > \frac{c_1}{x^{\frac{1}{2}}}\int_0^x A_1^2(x_1)\,dx_1 \tag{5.9}$$

for all sufficiently large x. Now write

$$x=y^{\frac{3}{2}},\quad A_1=y^{-\frac{1}{4}}\bar{A}(y), \tag{5.10}$$

so that, if (5.9) holds,

$$\frac{9c_1}{4}\int_0^y \bar{A}^2(y_1)\,dy_1 < \frac{d\bar{A}}{dy}-\frac{1}{4y}\bar{A} < \frac{d\bar{A}}{dy}, \tag{5.11}$$

for sufficiently large y. On integrating and absorbing the constant of integration into c_1,

$$c_2\left[\int_0^y \bar{A}^2\,dy_1\right]^{\frac{4}{3}} < \bar{A}^2(y) \tag{5.12}$$

for some constant $c_2(>0)$ and all sufficiently large y. A further integration then yields

$$-3\left[\int_0^y \bar{A}^2\,dy_1\right]^{-\frac{1}{3}} > c_2y+\text{const.} \tag{5.13}$$

for all sufficiently large y. This, however, implies a contradiction since the left-hand side is negative while the right-hand side tends to $+\infty$ with y.

As x increases, the solution of (5.2) probably ends when A_1 becomes infinite. If the singularity is algebraic and occurs at $x = \bar{x}$, we may verify that near $x = \bar{x}$

$$A_1 \approx \frac{4\lambda}{(\bar{x}-x)^{\frac{3}{2}}} \tag{5.14}$$

and a consistent expansion near $\bar{x} = x$ can be set up for A_1 with (5.14) as the leading term. Granting that this is the terminal structure of A_1 and noting that A_1 is large and positive when x is large and negative, it follows that the skin friction A_1 falls to a minimum before increasing again to infinity at $x = \bar{x}$.

We conclude that, when the mainstream is supersonic, a triple deck can consistently be set up but, like the original boundary layer, it ultimately breaks down so that at best it only serves to delay the singularity and it is still not possible to carry on to another boundary layer of conventional type. By contrast it is noted that, if the boundary layer starts to develop, spontaneously, a free-interaction zone from a clearly non-separated velocity profile, such as the Blasius profile, then separation is passed without any singularity whatsoever. There is plenty of experimental evidence to support the regularity of separation in free-interaction flows. It is not so clear exactly what happens in such natural separations as occur on bluff bodies. Is there always a dramatic change in the character of the flow at separation, paralleling that for subsonic flow, or does a free-interaction situation develop just upstream permitting reversed flow to be set up in a classical boundary layer?

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Appendix

We wish to solve

$$\frac{\partial^2 \tilde{\tilde{u}}_3}{\partial z^2} - \tfrac{1}{2}z^2 \frac{\partial \tilde{\tilde{u}}_3}{\partial x} - z\tilde{\tilde{v}}_3 = \frac{dp_3}{dx}, \quad \frac{\partial \tilde{\tilde{u}}_3}{\partial x} + \frac{\partial \tilde{\tilde{v}}_3}{\partial z} = 0 \tag{A 1}$$

subject to $\quad \tilde{\tilde{v}}_3(x,0) = \tilde{\tilde{u}}_3(-\infty, z) = \tilde{\tilde{u}}_3(x,\infty) = 0, \quad \tilde{\tilde{u}}_3(x,0) = L(x), \qquad$ (A 2)

where we suppose $L(x)$ is given and we wish to find $p_3(x)$ in terms of it. Differentiate (A 1) with respect to x and then with respect to z, when

$$\frac{\partial^4 \tilde{\tilde{v}}_3}{\partial z^4} - \tfrac{1}{2}z^2 \frac{\partial^3 \tilde{\tilde{v}}_3}{\partial z^2 \partial x} + \frac{\partial \tilde{\tilde{v}}_3}{\partial x} = 0 \tag{A 3}$$

on eliminating $\tilde{\tilde{u}}_3$. Now define the Fourier transform $\bar{v}_3$ of $\tilde{\tilde{v}}_3$ with respect to x by

$$\bar{v}_3(z;\omega) = \int_{-\infty}^{\infty} e^{-i\omega x}\,\tilde{\tilde{v}}_3(x,z)\,dx, \quad \tilde{\tilde{v}}_3 = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\,\bar{v}_3\,d\omega, \tag{A 4}$$

with similar definitions for $\bar{L}$ and $\bar{p}_3$. Then

$$\frac{d^4 \bar{v}_3}{dz^4} - \tfrac{1}{2}i\omega z^2 \frac{d^2 \bar{v}_3}{dz^2} + i\omega \bar{v}_3 = 0, \tag{A 5}$$

with $\quad \bar{v}_3 = 0, \quad \partial\bar{v}_3/\partial z = -i\omega\bar{L}, \quad \partial^3\bar{v}_3/\partial z^3 = \omega^2\bar{p}_3 \quad$ at $\quad z = 0 \quad$ and $\quad \bar{v}_3 \to 0 \quad$ (A 6) as $z \to \infty$. The general solution of (A 5) satisfying $\bar{v}_3 = 0$ at $z = 0$ is

$$\bar{v}_3 = C_1 \sum_{n=0}^{\infty} \frac{(n-\frac{5}{4})!\,(n-\frac{1}{2})!}{(4n+1)!}[8^{\frac{1}{4}}(0+i\omega)^{\frac{1}{4}}z]^{4n+1} + C_2 z^2 + C_3 \sum_{n=0}^{\infty} \frac{(n-\frac{3}{4})!\,n!}{(4n+3)!}[8^{\frac{1}{4}}(0+i\omega)^{\frac{1}{4}}z]^{4n+3}, \quad \text{(A 7)}$$

where C_1, C_2 and C_3 are constants and $(0+i\omega)^{\frac{1}{4}}$ is such that

$$(0+i\omega)^{\frac{1}{4}} = e^{\frac{1}{8}\pi i}|\omega|^{\frac{1}{4}} \quad \text{when } \omega \text{ is real and positive,}$$

$$(0+i\omega)^{\frac{1}{4}} = e^{-\frac{1}{8}\pi i}|\omega|^{\frac{1}{4}} \quad \text{when } \omega \text{ is real and negative.}$$

Further

$$\left.\begin{aligned} -i\omega\bar{L} &= C_1(-\tfrac{5}{4})!\,(-\tfrac{1}{2})!\,8^{\frac{1}{4}}(0+i\omega)^{\frac{1}{4}},\\ \omega^2\bar{p}_3 &= C_3(-\tfrac{3}{4})!\,8^{\frac{3}{4}}(0+i\omega)^{\frac{3}{4}}. \end{aligned}\right\} \quad \text{(A 8)}$$

The two series in (A 7) are convergent for all z but both have moduli which become exponentially large as $z \to \infty$. Hence in order to satisfy the condition $\bar{v}_3 \to 0$ as $z \to \infty$ a suitable combination of C_1, C_2 and C_3 must be taken. In fact we must have $C_1 = -C_3$ and then we may replace (A 7) by a single integral

$$\bar{v}_3 = \frac{C_1}{2i}\int_{-i\infty}^{i\infty} [8^{\frac{1}{4}}(0+i\omega)^{\frac{1}{4}}z]^{1+2s}\frac{(\frac{1}{2}s-\frac{5}{4})!\,(\frac{1}{2}s-\frac{1}{2})!}{(2s+1)!\sin s\pi}\,ds, \quad \text{(A 9)}$$

the integral passing just to the left of the origin of s. It may be verified that $\bar{v}_3$ is the same as (A 7), provided

$$C_2 = (-\tfrac{1}{4})!\,\pi 8^{\frac{1}{2}}(0+i\omega)^{\frac{1}{2}}C_1, \quad C_3 = -C_1 \quad \text{(A 10)}$$

and that then $\bar{v}_3 \to 0$ as $z \to \infty$. The first series of (A 7) is given by the residue of the simple poles when $s = 2n$ $(n = 0, 1, 2, \ldots)$, the second series by the residues at the simple poles when $s = 2n+1$ and the term C_2z^2 from the simple pole at $s = \frac{1}{2}$. Thus (A 1) has now been solved in principle. Of special interest here is the relation between $\bar{L}$ and $\bar{p}_3$ which follows from $C_1 = -C_3$ and is

$$\bar{L} = \frac{(-\frac{1}{4})!}{2(\frac{1}{4})!}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(0+i\omega)^{\frac{1}{2}}\bar{p}_3. \quad \text{(A 11)}$$

On inverting the Fourier transform it follows that

$$\bar{L}(x) = \frac{(-\frac{1}{4})!}{2^{\frac{3}{2}}(\frac{1}{4}!)}\int_{-\infty}^{x}\frac{p_3'(x_1)\,dx_1}{(x-x_1)^{\frac{1}{2}}}; \quad \text{(A 12)}$$

in turn this leads to (3.36), using (3.35) and (A 2), and noting that

$$A_1^2 + 8\alpha_1^2 x \to 0 \quad \text{as} \quad x \to -\infty.$$

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